Practice Questions for Midterm 2 - Math 1060Q - Fall 2013

The following is a selection of problems to help prepare you for the second midterm exam. Please note the following:

- there may be mistakes – email steven.pon@uconn.edu if you find one.
- the distribution of problems on this review sheet does not reflect the distribution that will be on the exam.
- learning math is about more than just memorizing the steps to certain problems. You have to understand the concepts behind the math. In order to test your understanding of the math, you may see questions on the exam that are unfamiliar, but that rely on the concepts you've learned. Therefore, you should concentrate on learning the underlying theory, not on memorizing steps without knowing *why* you're doing each step. (Some examples of "unfamiliar" problems can be found in the sample problems below, to give you an idea.)

Now trig functions. Angles can be measured in degrees or radians, and though we primarily use radians, it's worth knowing how to convert. More importantly, you should know the unit circle very well!

1. Convert $30°$ to radians.

Solution:

$$30° = \frac{\pi}{6} \text{ radians}$$

You can memorize a conversion formula, or just remember that 360 degrees equals 2π radians. $30 = \frac{360}{12}$, so $\frac{2\pi}{12} = \frac{\pi}{6}$ radians corresponds to 30 degrees.

2. Convert $\frac{\pi}{2}$ radians to degrees.

Solution:

$$\frac{\pi}{2} = 90°$$

3. Which of the following angles are coterminal? $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{-7\pi}{4}$

Solution: $\frac{\pi}{4}, \frac{9\pi}{4}$ and $-\frac{7\pi}{4}$ all correspond. To see this add $\frac{\pi}{4} + 2\pi = \frac{9\pi}{4}$. On the other hand, if we substract $\frac{\pi}{4} - 2\pi = -\frac{7\pi}{4}$.

4. What is an angle in the interval $[\pi, 2\pi]$ that corresponds to the angle $-\frac{\pi}{6}$ (i.e., is coterminal with $-\frac{\pi}{6}$)?

Solution: We can find the answer by the addition of 2π since our starting angle is below our desired range. Notice that $-\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$.

5. What is an angle in the interval $[-\pi, 0]$ that corresponds to the angle $\frac{5\pi}{3}$ (i.e., is coterminal with $-\frac{5\pi}{3}$)?

Solution: Just as above, $\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$. In this case we substract 2π in order to move into the desired range.

6. Draw the unit circle. Label the angles $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{3\pi}{4}, \pi, \frac{7\pi}{6}, \frac{7\pi}{4}, 2\pi, -\frac{\pi}{4}, -\frac{5\pi}{3}$. Label the coordinates of the points on the unit circle that correspond to those angles.
7. Find an angle in $[0, 2\pi)$ that is "coterminal" with the angle $\frac{65\pi}{6}$.

Solution: Since $\frac{5\pi}{6} = \frac{5\pi}{6} + \frac{60\pi}{6} = \frac{5\pi}{6} + 10\pi$
$\frac{5\pi}{6}$ is coterminal with the angle $\frac{65\pi}{6}$.

8. Find an angle in $[0, 2\pi)$ that is "coterminal" with the angle $\frac{14\pi}{5}$.

Solution: Since $\frac{14\pi}{5} = \frac{4\pi}{5} + \frac{10\pi}{5} = \frac{4\pi}{5} + 2\pi$
$\frac{4\pi}{5}$ is coterminal with the angle $\frac{14\pi}{5}$.

9. Find the "reference angle" for the angle $\frac{7\pi}{4}$. (The reference angle is the angle in $[0, \frac{\pi}{2}]$ formed between a given angle and the x-axis, so for example, the reference angle for $\frac{2\pi}{3}$ is $\frac{\pi}{3}$.)

Solution: The reference angle for $\frac{7\pi}{4}$ is $\frac{\pi}{4}$. (The angle $\frac{7\pi}{4}$ is in the fourth quadrant, you want to find how far "past" $\frac{7\pi}{4}$ you have to go to get to the x-axis – that's $\frac{\pi}{4}$.)

10. Find the "reference angle" for the angle $\frac{5\pi}{6}$.

Solution: The reference angle for $\frac{5\pi}{6}$ is $\frac{\pi}{6}$.

11. Find the "reference angle" for the angle $\frac{17\pi}{5}$.

Solution: The reference angle for $\frac{17\pi}{5}$ is $\frac{2\pi}{5}$.

Do you understand what the trigonometric functions are? How to calculate them, both using triangles or the unit circle?

12. Define sine, cosine, tangent, cosecant, secant, cotangent. What are their domain and range?

Solution: Let θ be the angle formed starting with the positive x-axis and going counterclockwise if θ is positive and clockwise if θ is negative. Then the corresponding "terminal point" on the unit circle has coordinates $(\cos(\theta), \sin(\theta))$.
The other trig functions are defined as follows:
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}, \sec(\theta) = \frac{1}{\cos(\theta)}, \csc(\theta) = \frac{1}{\sin(\theta)}$

Function	Domain	Range
$\sin(\theta)$	$(-\infty, \infty)$	$[-1, 1]$
$\cos(\theta)$	$(-\infty, \infty)$	$[-1, 1]$
$\tan(\theta)$	all real numbers except $\frac{n\pi}{2}$ for odd integers n	$(-\infty, \infty)$
$\cot(\theta)$	all real numbers except $n\pi$ for all integers n	$(-\infty, \infty)$
$\sec(\theta)$	all real numbers except $\frac{n\pi}{2}$ for odd integers n	$(-\infty, -1] \cup [1, \infty)$
$\csc(\theta)$	all real numbers except $n\pi$ for all integers n	$(-\infty, -1] \cup [1, \infty)$

For angles between 0 and $\frac{\pi}{2}$, these trig functions can be equivalently defined using right triangles. In a right triangle,
$\sin(\theta) = \frac{opposite}{hypotenus}, \cos(\theta) = \frac{adjacent}{hypotenus}$
$\tan(\theta) = \frac{opposite}{adjacent}, \cot(\theta) = \frac{adjacent}{opposite}$
$\sec(\theta) = \frac{hypotenuse}{adjacent}, \csc(\theta) = \frac{hypotenuse}{opposite}$.

13. What is $\sin(30°)$?

Solution: $\sin(30°) = \frac{1}{2}$

14. What is $\csc(\frac{5\pi}{6})$?

Solution: $\csc(\frac{5\pi}{6}) = \frac{1}{\sin(\frac{5\pi}{6})} = \frac{1}{\frac{1}{2}} = 2$

15. If $t = \frac{20\pi}{3}$, what are $\sin(t)$, $\csc(t)$, and $\cot(t)$?

Solution: $\sin(t) = \frac{\sqrt{3}}{2}$
$\csc(t) = \frac{1}{\sin(t)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$
$\cot(t) = \frac{\cos(t)}{\sin(t)} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$

16. If $\cot(t) = 1$ and t is in the interval $[\pi, 2\pi]$, then what is $\sin(t)$?

Solution: Since $\cot(t) = \frac{\cos(t)}{\sin(t)} = 1$
$t = \frac{5\pi}{4}$ is the angle in $[\pi, 2\pi]$ where $\cot(t) = 1$.
$\sin(t) = \sin(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

17. If $\cos(t) = -\frac{1}{2}$ and t is in the interval $[\pi, 2\pi]$, then what is $\tan(t)$?

Solution: $t = \frac{4\pi}{3}$ is the angle in $[\pi, 2\pi]$ where $\cos(t) = -\frac{1}{2}$.
$\tan(t) = \frac{\sin(t)}{\cos(t)} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$

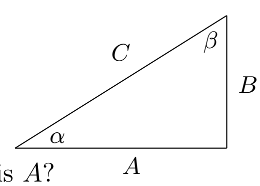
18. Given that $\cos(\theta) = \frac{2}{7}$ and θ is in quadrant IV, find $\sin(\theta)$.

Solution: Opposite side of θ is $\pm\sqrt{7^2 - 2^2} = \pm\sqrt{45}$
Since $\sin(\theta)$ is negative in quadrant IV,
so $\sin(\theta) = -\frac{\sqrt{45}}{7}$

19. Given that $\tan(\theta) = -\frac{3}{5}$ and θ is in quadrant II, find $\csc(\theta)$.

Solution: Hypotenuse side of θ is $\sqrt{3^2 + (-5)^2} = \sqrt{34}$
so $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{\frac{3}{\sqrt{34}}} = \frac{\sqrt{34}}{3}$

20. Use the triangle below to answer these questions.



(a) Say $\alpha = \frac{\pi}{4}$ and $B = 8$. What is A?
(b) Say $\beta = \frac{\pi}{6}$ and $A = 20$. What is C?
(c) Say $A = 5$ an $B = 10$. What is α?

Solution:

(a) $\tan(\frac{\pi}{4}) = \frac{8}{x} = 1$
Therefore, $x = 8$

(b) $\cos(\frac{\pi}{6}) = \frac{20}{x} = \frac{\sqrt{3}}{2}$
So $x = \frac{40}{\sqrt{3}}$

(c) $\tan(\alpha) = \frac{10}{5} = 2$
So $\alpha = \arctan(2)$

21. Is $\sin(\frac{19\pi}{20})$ closest to -1, 0, or 1?

Solution: Note that the angle $\frac{19\pi}{20}$ is almost π, but just barely less than π. On the unit circle, then, it's just above the x-axis, so the sine is close to 0.

What about going "backwards?" If you know the values of trig functions, can you work backwards to find out what the angle might be?

22. If $\sin(t) = \frac{\sqrt{2}}{2}$, what might t be? Give all possible solutions in $[-\pi, \pi]$.

Solution: $t = \frac{\pi}{4}, \frac{3\pi}{4}$

23. If $\sin t = 1$, what might t be? Give all possible solutions in $[0, 2\pi]$.

Solution: $t = \frac{\pi}{2}$

24. If $\cos t = \frac{1}{2}$, what might t be? Give all possible solutions in $[0, 2\pi]$.

Solution: $t = \frac{\pi}{3}, \frac{5\pi}{3}$

25. If $\tan t = -1$, what might t be? Give all possible solutions in $[0, 2\pi]$.

Solution: $t = \frac{3\pi}{4}, \frac{7\pi}{4}$

26. Find all values of t in the interval $[0, 2\pi]$ such that $\sin t = 0$.

Solution: $t = 0, \pi, 2\pi$

27. Find all values of t in the interval $[0, 2\pi]$ such that $\csc t = 2$.

Solution: i.e., $\frac{1}{\sin(t)} = 2$ so $\sin(t) = \frac{1}{2}$ and $t = \frac{\pi}{6}, \frac{5\pi}{6}$.

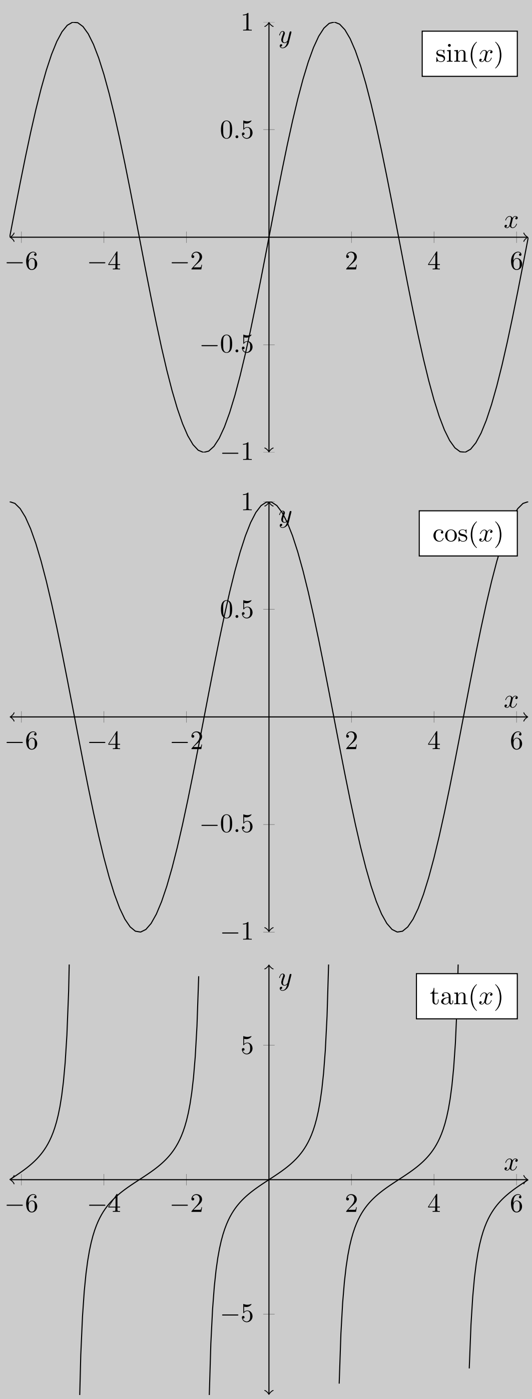
28. Find all values of t in the interval $[0, 2\pi)$ such that $\cot t = \sqrt{3}$.

Solution: i.e., $\frac{1}{\tan(t)} = \sqrt{3}$, so $\tan(t) = \frac{1}{\sqrt{3}}$. Then $t = \frac{\pi}{6}, \frac{7\pi}{6}$.

Like any other function, we sometimes want to graph trigonometric functions. Can you?

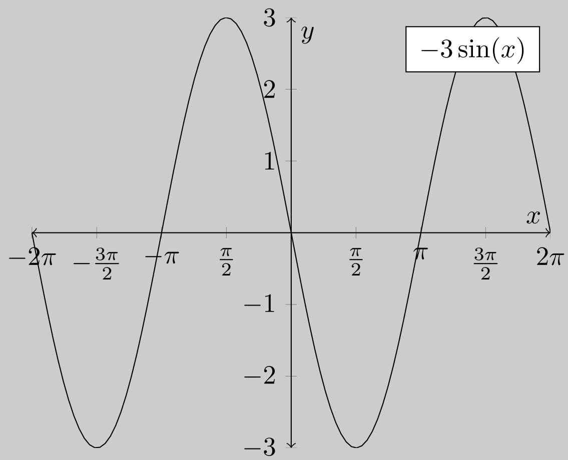
29. Graph $\sin(x)$, $\cos(x)$, and $\tan(x)$.

Solution:



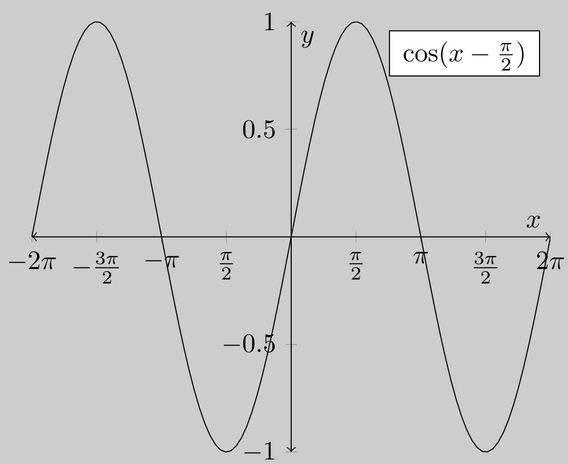
30. Graph $f(x) = -3\sin(x)$.

Solution:



31. Graph $g(x) = \cos(x - \frac{\pi}{2})$. Can you find another function that has this same graph?

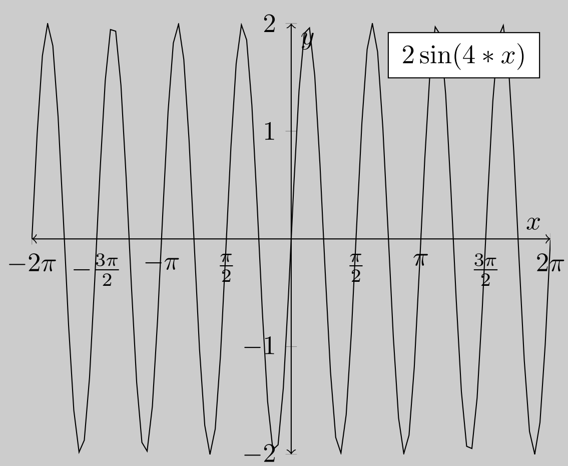
Solution:



This is the same as $\sin(x)$.

32. Graph $h(x) = 2\sin(4x)$.

Solution:

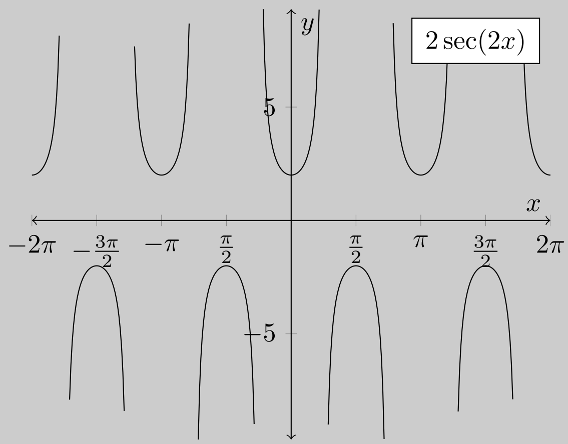


33. Where does $\csc(x)$ have a vertical asymptote?

Solution: The function $\csc(x)$ is the same as $\frac{1}{\sin(x)}$, so it is undefined (and has a vertical asymptote) whenever $\sin(x) = 0$. That's at multiples of π: $0, \pi, 2\pi$, etc.

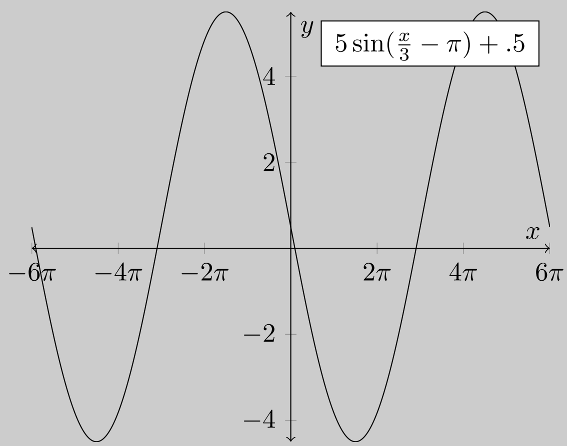
34. Graph $2\sec(2x)$.

Solution:



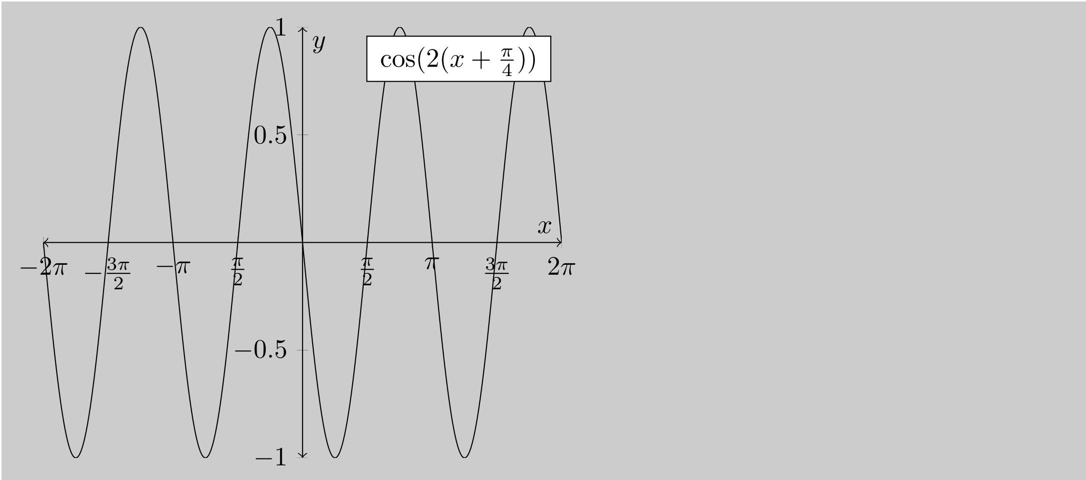
35. Graph $5\sin(\frac{x}{3} - \pi) + .5$.

Solution:



36. Graph $\cos(2(x + \frac{\pi}{4}))$.

Solution:



Can you solve equations that involve trigonometric functions?

37. Find all solutions to $2\sin(x)+1=0$ in $[0,2\pi)$.

Solution:

$$\begin{aligned} 2\sin(x)+1&=0 \\ \sin(x)&=-\frac{1}{2} \\ x&=\frac{7\pi}{6},\frac{11\pi}{6} \end{aligned}$$

38. Find all solutions to $3\cos x=3$.

Solution:

$$\begin{aligned} 3\cos x&=3 \\ \cos x&=1 \\ x&=0,2\pi,4\pi,\ldots \\ x&=2\pi k \end{aligned}$$

39. Find all solutions to $3\sin x-4=\sin x-2$ in $[0,2\pi)$.

Solution:

$$\begin{aligned} 3\sin x-4&=\sin x-2 \\ 2\sin x&=2 \\ \sin x&=1 \\ x&=\frac{\pi}{2} \end{aligned}$$

40. Find all solutions to $4\cos^2 x-1=0$ in $[0,2\pi)$.

Solution:

$$\begin{aligned} 4\cos^2 x - 1 &= 0 \\ \cos^2 x &= \frac{1}{4} \\ \cos x &= \pm\frac{1}{2} \\ x &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

41. Find all solutions to $\cos(2x) = \frac{1}{2}$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} \cos(2x) &= \frac{1}{2} \\ 2x &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ x &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

42. Find all solutions to $\tan^2(x) = 3$.

Solution:

$$\begin{aligned} \tan^2(x) &= 3 \\ \tan(x) &= \pm\sqrt{3} \\ x &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} + 2\pi k \end{aligned}$$

43. Find all solutions to $\sin x + \sqrt{2} = -\sin x$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} \sin x + \sqrt{2} &= -\sin x \\ 2\sin x &= -\sqrt{2} \\ \sin x &= -\frac{\sqrt{2}}{2} \\ x &= \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

44. Find all solutions to $2\cos(3x - 1) = 0$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} 2\cos(3x-1) &= 0 \\ \cos(3x-1) &= 0 \\ 3x - 1 &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \\ x &= \frac{\pi+2}{6}, \frac{3\pi+2}{6}, \frac{5\pi+2}{6}, \frac{7\pi+2}{6}, \frac{9\pi+2}{6}, \frac{11\pi+2}{6} \end{aligned}$$

45. Find all solutions to $\sec x = 2\cos x$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} \frac{1}{\cos x} &= 2\cos x \\ 1 &= 2\cos^2 x \\ \frac{1}{2} &= \cos^2 x \\ \cos x &= \pm\frac{\sqrt{2}}{2} \\ x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

46. Find all solutions to $2\sin^2 x - \sin x - 1 = 0$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} 2\sin^2 x - \sin x - 1 &= 0 \\ (2\sin x + 1)(\sin x - 1) &= 0 \\ \sin x &= -\frac{1}{2}, 1 \\ x &= \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

47. Find all solutions to $\sin x \cos x + \cos x = 0$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} \sin x \cos x + \cos x &= 0 \\ \cos x\,(\sin x + 1) &= 0 \\ \cos x &= 0 \text{ or } \sin x = -1 \\ x &= \frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$

48. Find all solutions to $\sin(2x) = -\cos(2x)$.

Solution:

$$\begin{aligned} \sin(2x) &= -\cos(2x) \\ \frac{\sin 2x}{\cos 2x} &= -1 \\ \tan 2x &= -1 \\ 2x &= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4} + 2\pi k, \ldots \\ x &= \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8} + \pi k \end{aligned}$$

Sometimes you need to use a trigonometric identity to help solve an equation. You'll need to use trig identities in the following problems.

49. Find all solutions to $2\cos^2 x + 3\sin x = 3$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} 2\cos^2 x + 3\sin x &= 3 \\ 2\left(1 - \sin^2 x\right) + 3\sin x - 3 &= 0 \\ 2\sin^2 x - 3\sin x + 1 &= 0 \\ (2\sin x - 1)(\sin x - 1) &= 0 \\ \sin x = \frac{1}{2} \text{ or } \sin x &= 1 \\ x &= \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6} \end{aligned}$$

50. Find all solutions to $\sin(2x) = \sqrt{3}\cos x$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} \sin(2x) &= \sqrt{3}\cos x \\ 2\sin x\cos x &= \sqrt{3}\cos x \\ 2\sin x\cos x - \sqrt{3}\cos x &= 0 \\ \cos x\left(2\sin x - \sqrt{3}\right) &= 0 \\ \cos x = 0 \text{ or } \sin x &= \frac{\sqrt{3}}{2} \\ x &= \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2} \end{aligned}$$

51. Find all solutions to $2\sqrt{2}\sin(2t) - 2\tan(2t) = 0$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} 2\sqrt{2}\sin(2t) - 2\tan(2t) &= 0 \\ 2\sqrt{2}\sin(2t)\cos(2t) - 2\sin(2t) &= 0 \\ \sin(2t)\left(2\sqrt{2}\cos(2t) - 2\right) &= 0 \\ \sin(2t) = 0 \text{ or } \cos(2t) &= \frac{\sqrt{2}}{2} \\ 2t = 0, \pi, 2\pi, 3\pi, \ldots \text{ or } 2t &= \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \ldots \\ t &= 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8} \end{aligned}$$

52. Find all solutions to $2\cot^2 x + \csc^2 x - 2 = 0$ in $[0, 2\pi)$.

Solution:

$$\begin{aligned} 2\cot^2 x + \csc^2 x - 2 &= 0 \\ 2\cot^2 x + \left(1 + \cot^2 x\right) - 2 &= 0 \\ 3\cot^2 x - 1 &= 0 \\ \cot x &= \pm\frac{\sqrt{3}}{3} \\ x &= \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \end{aligned}$$

53. Find all solutions to $\cos(2x) = \cos x$ in $[0, 2\pi]$.

Solution:

$$\begin{aligned} \cos(2x) &= \cos x \\ 2\cos^2 x - 1 &= \cos x \\ 2\cos^2 x - \cos x - 1 &= 0 \\ (2\cos x + 1)(\cos x - 1) &= 0 \\ \cos x &= -\frac{1}{2} \text{ or } \cos x = 1 \\ x &= 0, \frac{2\pi}{3}, \frac{4\pi}{3} \end{aligned}$$

Just like with other functions we've studied, trigonometric functions have inverses. Well, partial inverses (since trig functions aren't one-to-one). Do you know what the inverse trig functions are? Do you know their domain and range? Can you compute them for some values? Can you graph them?

54. What is $\arccos \frac{1}{2}$?

Solution: $\theta = \frac{\pi}{3}$

55. What is $\arcsin \frac{-\sqrt{2}}{2}$?

Solution: $\theta = \frac{-\pi}{4}$

56. What is $\arctan 0$?

Solution: Let $\arctan 0 = \theta$.
Note: θ must be in $(-\pi/2, \pi/2)$.
$\tan(\theta) = 0$. Therefore, $\theta = 0$.

57. What is $\sin^{-1}(\sin(\frac{3\pi}{2}))$?

Solution: Let $\sin^{-1}(\sin(\frac{3\pi}{2})) = \theta$.
Note: θ must be in $[-\pi/2, \pi/2]$.
since $\frac{3\pi}{2}$ is coterminal with $\frac{-\pi}{2}$
$\sin^{-1}(\sin(\frac{3\pi}{2})) = \sin^{-1}(\sin(\frac{-\pi}{2})) = \frac{-\pi}{2}$

Answer: $\theta = \frac{-\pi}{2}$

58. What is $\tan(\arctan(-3))$?

Solution: Note: $\tan(\arctan(y)) = y$ if y is in $(-\infty, \infty)$. Since -3 is in $(-\infty, \infty)$
Answer: $\tan(\arctan(-3))$=-3

59. What is $\arccos(\cos(-\frac{4\pi}{9}))$?

Solution: Note: $\arccos(\cos(\theta)) = \theta$ if θ is in the range of the arccosine function, i.e. $[0, \pi]$.
Since cosine function is an even function $\cos(-\frac{4\pi}{9}) = \cos(\frac{4\pi}{9})$
$\arccos(\cos(-\frac{4\pi}{9}))$=$\arccos(\cos(\frac{4\pi}{9}))$=$\frac{4\pi}{9}$
Answer: $\frac{4\pi}{9}$

60. What is $\tan(\sin^{-1}(\frac{\sqrt{2}}{2}))$?

Solution: Let $\sin^{-1}(\frac{\sqrt{2}}{2})$=$\theta$.
Then $\sin\theta$=$\frac{\sqrt{2}}{2}$.
Using SOH,CAH,TOA
O=$\sqrt{2}$, H=2, A=$\sqrt{2}$
and $\tan\theta$=$\frac{\sqrt{2}}{\sqrt{2}}$=1
So $\tan(\sin^{-1}(\frac{\sqrt{2}}{2})$=$\tan\theta$=1
Answer:1

61. What is $\tan(\arccos(\frac{5}{13}))$?

Solution: Let $\arccos(\frac{5}{13})$=θ.
Then $\cos\theta$=$\frac{5}{13}$.
Using SOH,CAH,TOA
A=5, H=13, O=12
and $\tan\theta$=$\frac{12}{5}$
So $\tan(\arccos(\frac{5}{13})$=$\tan\theta$=$\frac{12}{5}$
Answer:$\frac{12}{5}$

62. Express a solution to the equation $\tan(x - 3) = 5$ using inverse trigonometric functions.

Solution:

$$\arctan(5) = x - 3$$
$$x = \arctan(5) + 3$$

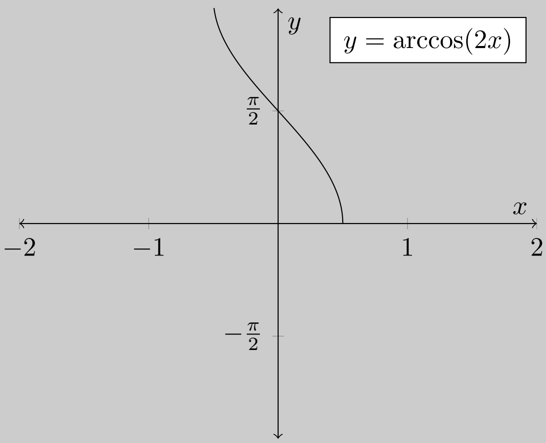
63. Express a solution to the equation $\sin(4x) + 1 = \frac{2}{3}$ using inverse trigonometric functions.

Solution:

$$\sin(4x) = -\frac{1}{3}$$
$$4x = \arcsin(-\frac{1}{3})$$
$$x = \frac{1}{4}\arcsin(-\frac{1}{3})$$

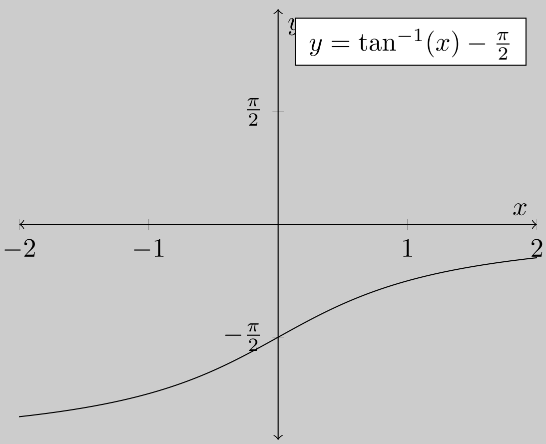
64. Graph the function $\cos^{-1}(2x)$.

Solution:



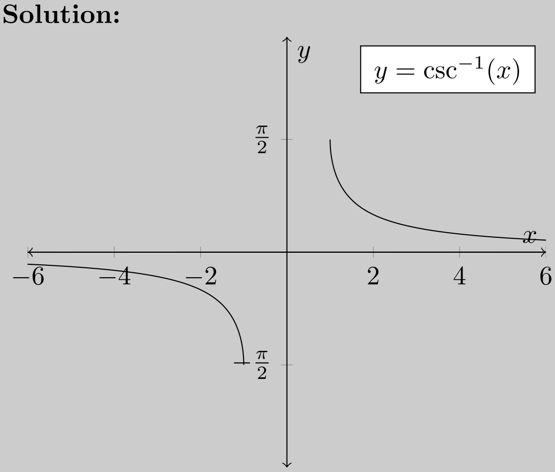
65. Graph the function $\arctan(x) - \frac{\pi}{2}$.

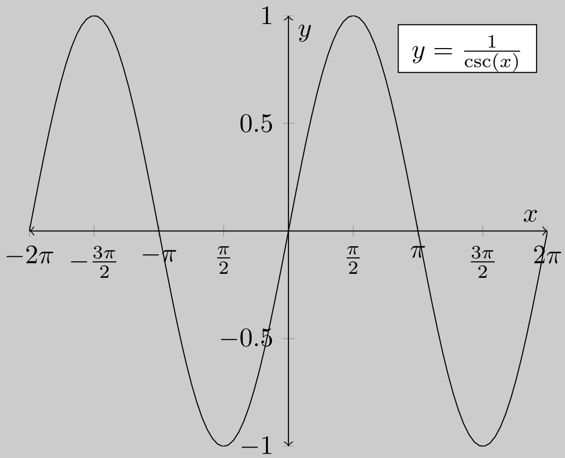
Solution:



66. Graph the functions $\csc^{-1}(x)$ and $\frac{1}{\csc(x)}$.

Solution:





Note that $\frac{1}{\csc(x)} = \frac{1}{\frac{1}{\sin(x)}} = \sin(x)$.

Can you apply your knowledge to real-world applications?

67. How long a ladder do you need if you want to reach a window that's 20 feet off the ground? You're on uneven terrain, so leaning a ladder any steeper than 60 degrees is unsafe.

Solution: The ladder would be the hypotenuse of a triangle that has height 20 feet and is leaned at an angle of 60 degrees. Then $\sin(60°) = \frac{20}{x}$, where x is the length of the ladder. Therefore, $\frac{\sqrt{3}}{2} = \frac{20}{x}$, so $x = \frac{40}{\sqrt{3}}$.

68. An equilateral triangle is circumscribed around a circle of radius 5 (thus, the center of the triangle and the center of the circle are at the same point). What is the area of the triangle?

Solution: Equilateral triangles have angles of size $\frac{\pi}{3}$. If the triangle is circumscribed around the circle, that means the distance from the center of the triangle to one of its edges is the same as the radius of the circle, so it's 5. Then you can make a right triangle, with a line connecting the center of the circle to the edge of the triangle and another line connecting the center of the circle to a vertex of the triangle. This cuts one of the angles of the equilateral triangle in half, so the right triangle has an angle of $\frac{\pi}{6}$. The opposite side to that angle has length 5, so $\sin(\frac{\pi}{6}) = \frac{5}{x}$, or in other words, $x = 10$ is the length of the line connecting the center of the circle to a vertex of the triangle. Therefore, the height of the equilateral triangle is 15 (draw a picture to help see why that's the case). Then, let's find the length of one side of the equilateral triangle using $\cos(\frac{\pi}{6}) = \frac{z}{10}$, where z is half the length of one side of the equilateral triangle. That means that $z = 5\sqrt{3}$, so the length of the base of the equilateral triangle is $10\sqrt{3}$. Therefore, the area of the triangle is $\frac{15 \cdot 10\sqrt{3}}{2}$.

69. A rope hanging straight down from a pole is 4 feet longer than the pole (that is, the rope hangs straight down and there's 4 feet of extra rope lying on the ground). When the rope is picked up and stretched taut, it hits the ground 8 feet away from the pole. How tall is the pole?

Solution: Draw a picture of the situation. Call the length of the pole x. Then we have a triangle with legs x and 8, with hypotenuse $x + 4$. Using the pythagorean theorem we have

$$x^2 + 8^2 = (x + 4)^2$$

Now we can solve this equation:

$$x^2 + 64 = x^2 + 8x + 16$$

$$64 = 8x + 16$$
$$8 = x + 2$$
$$6 = x$$

70. The number of hours of daylight in a day can be modeled as a sinusoidal curve. Assume that the longest day of the year is June 21, which has 15.2 hours of daylight, and the shortest day of the year is December 21, which has 9.1 hours of daylight (this is close to the real numbers in CT). Let's start the clock on January 1, and say Jan 1 corresponds to $t = 0$. Find a sinusoid to model the number of hours of daylight as a function of the day of the year. Use this to estimate the number of hours of daylight there will be tomorrow. The numbers in this problem are messy, so it's not a great exam problem...but it's kind of fun.

Solution: If sunlight ranges from 15.2 to 9.1, then there is a difference of 15.2 - 9.1 = 6.1 between our max and min, so the amplitude is $\frac{6.1}{2} = 3.05$. The midline, then, is $9.1 + 3.05 = 12.15$. The period should be 365, of course (yes, in real life it would not be exactly 365, more like 365.24...but it's close enough for us). There is a shift, which might be easiest to see by counting backwards – the lowest point on the sine curve should be Dec 21, 10 days before Jan 1 (which is $t = 0$). Thus, the high point is halfway between $t = -10$ and $t = 355$, a year later, so the high point is approximately 172. The curve should pass the midline exactly halfway between the low point and the high point; halfway between -10 and 172 is 81. Thus, the curve should be shifted to the right by 81. A curve we could use is $3.05\sin(\frac{2\pi}{365}(t-81)) + 12.15$. As this is being written, it's late October, almost Halloween, which is the 304th day of the year (so would correspond to $t = 303$ in our model). The predicted number of hours of sunlight on October 31 is then $3.05\sin(\frac{2\pi}{365}(303-81)) + 12.15 = 10.23$. (The actual number of hours of sunlight is 10.4...not bad!)

71. A 27 inch (diameter) bicycle wheel turns at 120 revolutions per minute. There is a nail in the wheel. Find a function to model the height of the nail above the ground at any point in time, assuming we start the clock when the nail hits the pavement.

Solution: Minimum value is 0, maximum value is 27, so midline is 13.5 and amplitude is 13.5. 120 revolutions per minute means one period is .5 seconds. For variety, let's graph this one as a cosine function. If we use a negative cosine function, no shift is needed. Thus, an equation for the height of the nail is $-13.5\cos(\frac{2\pi}{.5}t) + 13.5$ inches at t seconds.

72. A weight hangs at the end of a spring, causing it to bounce up and down. At its lowest point, the weight sits 60 cm above the ground, and at its highest point, it reaches 100 cm above the ground. Every 10 seconds, the weight bounces up and down 4 times. The height of the weight above the ground can be modeled by a sine curve. Supposing the weight starts from its lowest point at time $t = 0$ seconds, find a sinusoid to model the height of the weight at time t.

Solution: This sine curve goes between 60 and 100, so the midline is 80 and the amplitude is 20. It goes through 4 periods in 10 seconds, so one period is $\frac{10}{4} = 2.5$ seconds. To model this as a sine curve, we need to apply a shift – the graph passes the midline at $\frac{1.25}{2} = .625$ seconds. Thus, a function to model this is $20\sin(\frac{2\pi}{2.5}(t - .625)) + 80$.

73. A plane is flying at an altitude of 36,000 feet. Looking down, the pilot sees a ship at an angle of depression of 30 degrees. He sees a submarine at an angle of depression of 60 degrees in the same direction. How far apart are the ship and submarine?

Solution: Draw a picture of the situation. We get two right triangles, Triangle A with the angle 30° with opposite side 36,000 and Triangle B with the angle 60° with opposite side

36,000. To find the distance between the ship and submarine we need find the difference between the adjacent side lengths of these two triangles.

For Triangle A, call the adjacent side a. Then we have

$$\begin{aligned} \tan(30^\circ) &= \frac{36000}{a} \\ \frac{1}{\sqrt{3}} &= \frac{36000}{a} \\ \sqrt{3} &= \frac{a}{36000} \\ 36000\sqrt{3} &= a \end{aligned}$$

For Triangle B, call the adjacent side b. Then we have

$$\begin{aligned} \tan(60^\circ) &= \frac{36000}{b} \\ \sqrt{3} &= \frac{36000}{b} \\ \frac{1}{\sqrt{3}} &= \frac{b}{36000} \\ \frac{36000}{\sqrt{3}} &= b \end{aligned}$$

Now the distance between the ship and submarine is $a - b$ which is:

$$36000\sqrt{3} - \frac{36000}{\sqrt{3}} = 360000(\sqrt{3} - \frac{1}{\sqrt{3}}) \text{ feet}$$

Here are a few additional problems that didn't make it into any of the above sections.

74. Calculate $\sin(\frac{\pi}{3} - \frac{\pi}{4})$ without using a calculator.

Solution: Use the difference identity for sine:

$$\begin{aligned} \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) &= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

75. Simplify the expression using identities as needed: $(1 - 2\cos^2 x + \cos^4 x)$.

Solution: It helps to first factor this equation:

$$1 - 2\cos^2 x + \cos^4 x = (1 - \cos^2 x)^2$$

Then we see that we can apply the pythagorean identity $\sin^2 x = 1 - \cos^2 x$ to obtain:

$$(1 - \cos^2 x)^2 = (\sin^2 x)^2 = \sin^4 x.$$

76. Calculate $\cos(\frac{7\pi}{8})$ without using a calculator.

Solution: Use the half angle formula for cosine:

$$\begin{aligned}\cos^2\left(\frac{7\pi}{8}\right) &= \frac{1+\cos(2\cdot\frac{7\pi}{8})}{2}\\ &= \frac{1+\cos(\frac{7\pi}{4})}{2}\\ &= \frac{1+\frac{\sqrt{2}}{2}}{2}\\ &= \frac{2+\sqrt{2}}{4}\end{aligned}$$

Now, taking the square root of both sides we get

$$\cos\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{2+\sqrt{2}}{4}}$$

but since the angle $\frac{7\pi}{8}$ is in the second quadrant, we know that the cosine of this angle should be negative. Thus,

$$\cos\left(\frac{7\pi}{8}\right) = -\sqrt{\frac{2+\sqrt{2}}{4}}.$$

77. Simplify $\frac{3}{\tan^2 x+1}$.

Solution: Use the pythagorean identity $\tan^2 x + 1 = \sec^2 x$ and $\sec x = \frac{1}{\cos x}$ and we see that

$$\frac{3}{\tan^2 x+1} = \frac{3}{\sec^2 x} = 3\cos^2 x.$$

78. True or false? $3\cos x + 3\sin x \tan x = 3\sec x$.

Solution: True.

We need to start with $3\cos x + 3\sin x\tan x$ and transform it into $3\sec x$. We use a series of trig identities:

$$\begin{aligned}3\cos x + 3\sin x\tan x &= 3\cos x + 3\sin x\frac{\sin x}{\cos x}\\ &= 3\cos x + \frac{3\sin^2 x}{\cos x}\\ &= \frac{3\cos^2 x + 3\sin^2 x}{\cos x}\\ &= \frac{3(\cos^2 x+\sin^2 x)}{\cos x}\\ &= \frac{3(1)}{\cos x}\ \text{(pythagorean identity)}\\ &= \frac{3}{\cos x}\\ &= 3\sec x\end{aligned}$$

79. True or false? $\frac{3\cot^3 x}{\csc x} = 3\cos x(\csc^2 x - 1)$

Solution: True.

We would like to start with $\frac{3\cot^3 x}{\csc x}$ and transform it into $3\cos x(\csc^2 x - 1)$. We'll use a series of trig identities:

$$\begin{aligned}
\frac{3\cot^3 x}{\csc x} &= 3\cot^2 x\left(\frac{\cot x}{\csc x}\right)\\
&= 3(\csc^2 x - 1)\left(\frac{\cot x}{\csc x}\right) \text{ (pythagorean identity)}\\
&= 3(\csc^2 x - 1)\left(\frac{\cos x}{\sin x}\sin x\right) \text{ switching to sine and cosine}\\
&= 3\cos x(\csc^2 x - 1)
\end{aligned}$$

80. Show work to prove that $\frac{3\sec\theta - 3}{1-\cos\theta} = 3\sec\theta$.

Solution: We need to start with $\frac{3\sec\theta - 3}{1-\cos\theta}$ and transform it into $3\sec\theta$. We use a series of trig identities:

$$\begin{aligned}
\frac{3\sec\theta - 3}{1-\cos\theta} &= 3\frac{\sec\theta - 1}{1-\cos\theta}\\
&= 3\frac{\frac{1}{\cos\theta} - 1}{1-\cos\theta}\\
&= 3\frac{\frac{1-\cos\theta}{\cos\theta}}{1-\cos\theta}\\
&= 3\frac{1}{\cos\theta}\\
&= 3\sec\theta
\end{aligned}$$